On the ill-posedness and stability of the relativistic heat equation

A.L. García Perciante y O.R. Reula

Casa abierta al tiempo
UNIVERSIDAD AUTÓNOMA METROPOLITANA
Unidad Cuajimalpa

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The System

The setting is a space-time manifold (M, g)
and a unit, time-like vector field u^a

The variable is a scalar field T

There is a dependent, auxiliary vector field q^a, perpendicular to u^a

$$
\begin{aligned}
u^a \nabla_a T &= \nabla^a q^a \\
\sigma q_a &= h_a{}^b \nabla_b T \qquad\qquad (h_a{}^b = \delta_a{}^b + u_a u^b)
\end{aligned}
$$

Assume: Fourier law is valid at a surface perpendicular to the *fluid* four velocity

One Dimensional Case

Heat Equation

$$\sigma \frac{dT}{dt} = \partial_x^2 T$$

$$T = T_0 e^{i(\omega t + kx)} \qquad \omega = \frac{ik^2}{\sigma}$$

Stable and decaying (parabolic behavior)
Well posed Initial Value Problem for that surface.

If we pretend to use another hypersurface:

$$u^a = \gamma(t^a + \beta^a)$$

$$\sigma\gamma(\partial_t + \beta\partial_x)T = \beta^2\gamma^2\partial_t^2 + 2\gamma^2\beta\partial_t\partial_x + \gamma^2\partial_x^2$$

$$i\sigma(\omega + \beta k)T = -\gamma(\beta^2\omega^2 + 2\beta\omega k + k^2)T$$

k fixed, $\beta \to 0$

$$\omega_+ \to \frac{ik^2}{\sigma}$$

$$\omega_- \to \frac{-i\sigma + 2\beta\gamma k}{\beta^2\gamma}$$

One dimensional case

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$$\sigma\gamma(\partial_t + \beta\partial_x)T = \beta^2\gamma^2\partial_t^2 + 2\gamma^2\beta\partial_t\partial_x + \gamma^2\partial_x^2$$

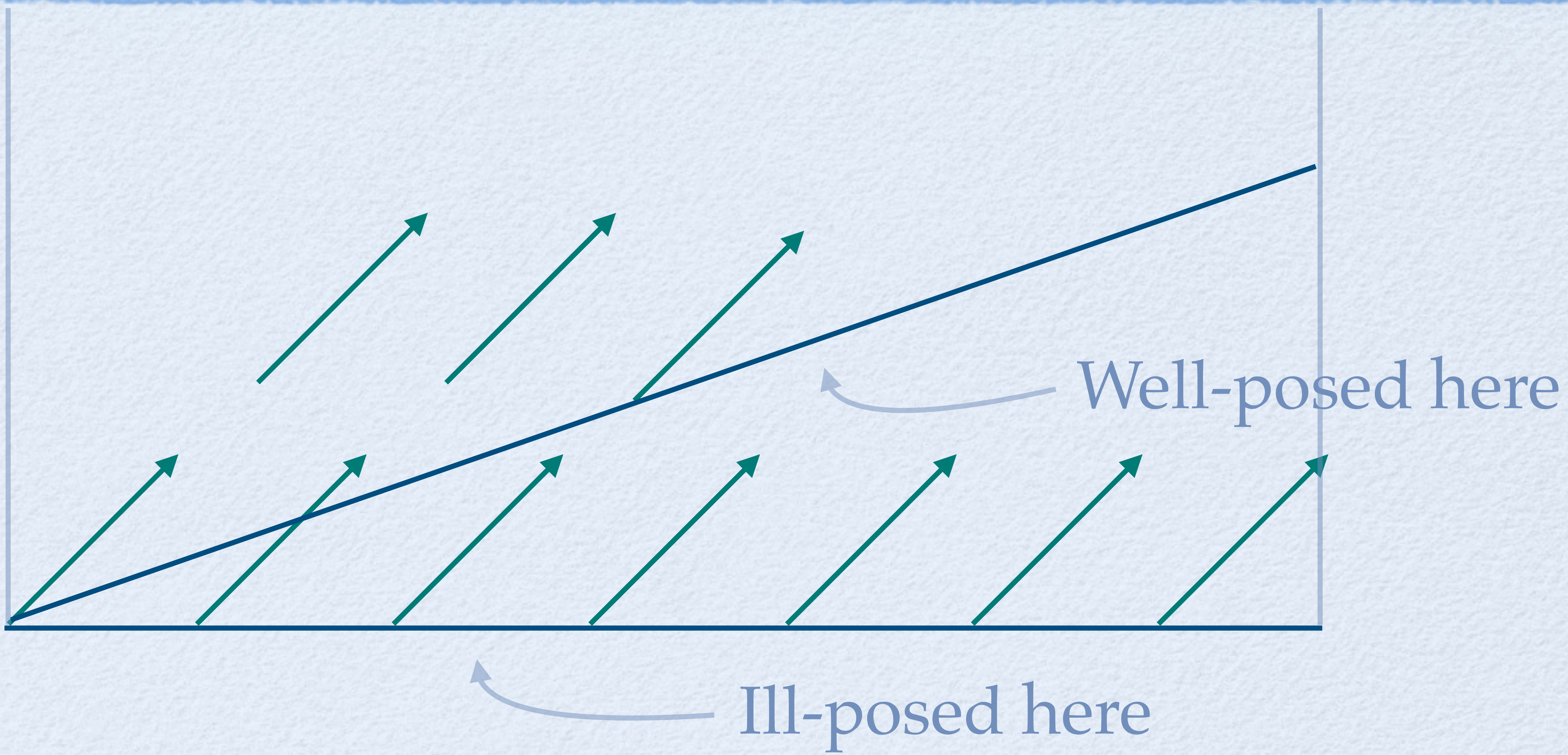
$$i\sigma(\omega + \beta k)T = -\gamma(\beta^2\omega^2 + 2\beta\omega k + k^2)T$$

β fixed, $k \to \infty$

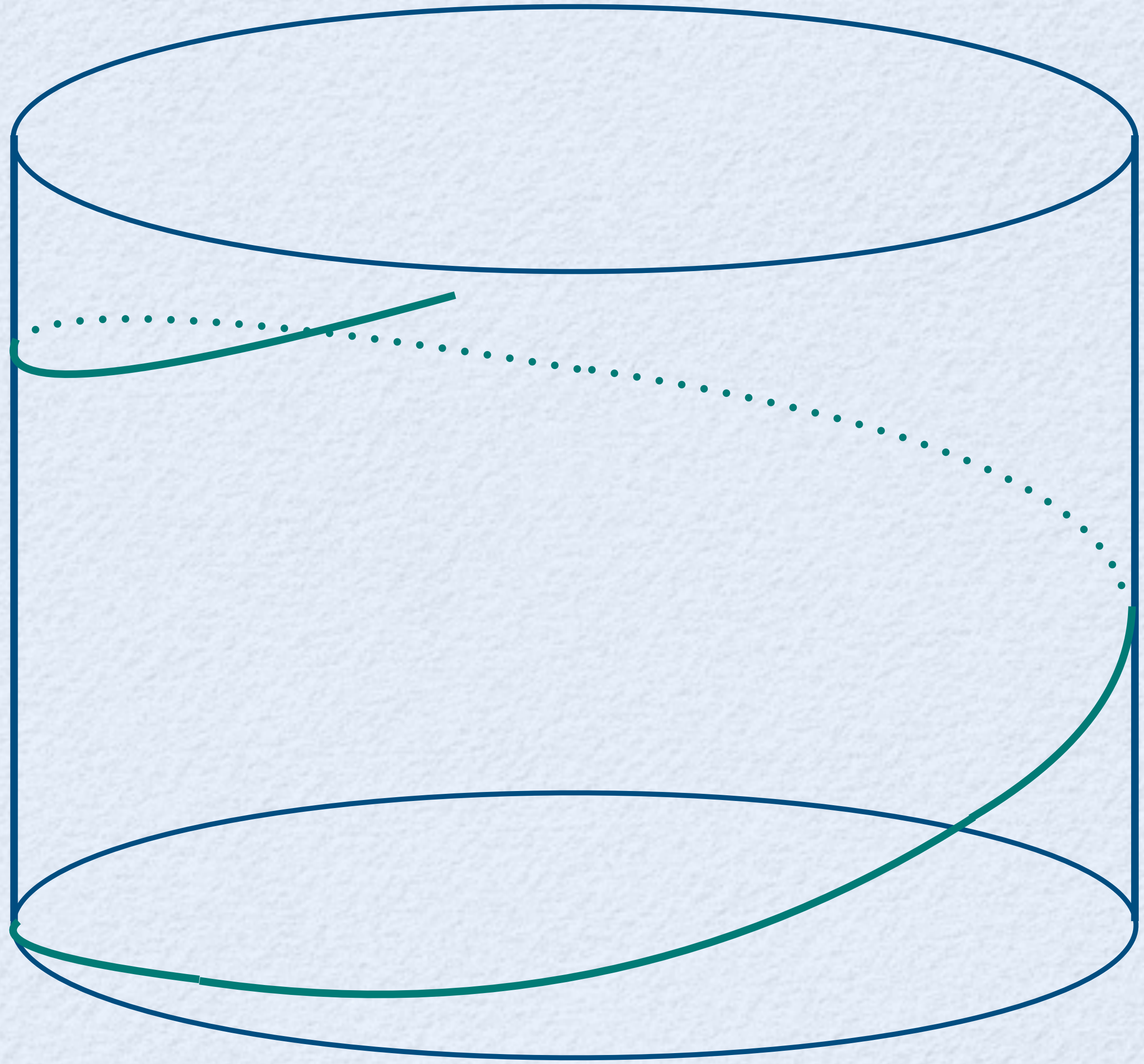
$$\omega_\pm = -\frac{k}{\beta} \pm \sqrt{\frac{\sigma}{\beta^3}}\frac{1}{\gamma}(1+i)\sqrt{k}$$

In more dimensions growth as k

Characteristic Surface: Simple Case



Characteristic surface: wrapped



Generic case: more dimensions

In the case the vector u^a is not surface forming we even have a local problem, we can not, even locally, find a hypersurface where the Cauchy problem is well posed.

The infinite propagation speed in the parabolic theories accepts only Cauchy data in a characteristic surface.

This is the reason why we need second order, hyperbolic, theories.

Making it Hyperbolic

$$\begin{aligned} u^a \nabla_a T &= \nabla^a q_a \\ \epsilon u^b \nabla_b q_a + q_a &= \frac{1}{\sigma} h_a{}^b \nabla_b T \qquad\qquad (h_a{}^b = \delta_a{}^b + u_a u^b) \end{aligned}$$

Hyperbolic when:

$$\sigma\epsilon = v_{ss}^{-2} > \beta^2$$

Tension between ϵ small and Einstein Causality

Question: How big does ϵ (or β^2) has to be according to the lack of surface forming property of u^a?

Making it hyperbolic

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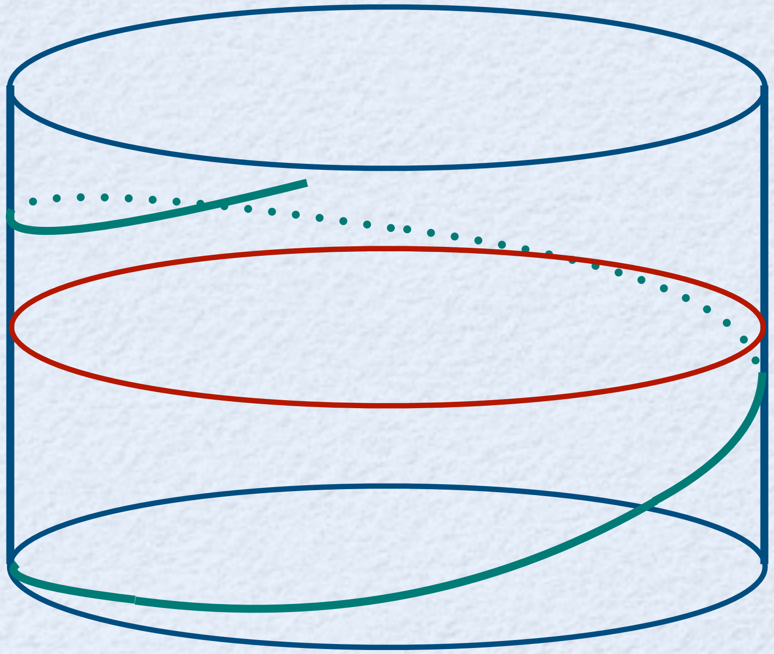
$$w_{abc} := u_{[a}\nabla_b u_{c]} \quad ?$$



Seems to be a global property of the fluid congruence.

Conclusions

- Parabolic equations do not work for lack of characteristic flats to form hyper surfaces.
- Local-global aspects.
- Hyperbolizations need to have slow enough velocities (tension with fast decay?)
- If propagations speeds are slow enough, then the problem is back to GR and the light cone structure.
- Mostly a mathematical question.

THANK YOU
FOR YOUR ATTENTION